Matrix trace Wielandt inequalities with statistical applications

Shuangzhe Liu\textsuperscript{a,∗}, Changyu Lu\textsuperscript{b}, Simo Puntanen\textsuperscript{c}

\textsuperscript{a}Faculty of Information Sciences and Engineering, University of Canberra, Canberra, ACT 2601, Australia
\textsuperscript{b}Financial Research Center, Shanghai Finance University, Shanghai 201209, China
\textsuperscript{c}Department of Mathematics and Statistics, University of Tampere, FI-33014, Finland

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\textbf{ABSTRACT}

The vector correlation coefficient and other measures of association play a very important role in statistics and especially in multivariate analysis. In this paper a new measure of association is proposed and its upper bound is presented by using a matrix trace Wielandt inequality. Also given are relevant results involving Wishart matrices widely used in multivariate analysis, and especially a new alternative for the relative gain of the covariance adjusted estimator of a vector of parameters.

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\textbf{1. Introduction}

It seems that the Wielandt inequality (WI) in the vector case was introduced by Bauer and Householder (1960) due to a private communication from Wielandt; see Drury et al. (2002, Section 2). Let $A$ be a positive definite symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n > 0$, and let $x$ and $y$ be two nonnull real vectors satisfying $x^\prime y = 0$. Then

$$\frac{(x^\prime Ay)^2}{x^\prime Ax \cdot y^\prime Ay} \leq \frac{(\lambda_1 - \lambda_n)^2}{\lambda_1 + \lambda_n}. \quad (1.1)$$

We will refer to (1.1) as the "WI". The first appearance of (1.1) in a statistical context seems to be by Eaton (1976). Let the random vector $h$ have the covariance matrix $A$; then the maximum of the squared correlation

$$\max_{x,y : x^\prime y = 0} \text{corr}^2(x, y) = \max_{x,y : x^\prime y = 0} \frac{(x^\prime Ay)^2}{x^\prime Ax \cdot y^\prime Ay} = \frac{(\lambda_1 - \lambda_n)^2}{\lambda_1 + \lambda_n}. \quad (1.2)$$

It is known that the WI can be viewed as a constrained version of the Cauchy–Schwarz inequality (CSI), which links with the Frucht–Kantorovich inequality (FKI) in a nice way. We remind the reader about the FKI which can be expressed as follows:

$$\frac{x^\prime Ax \cdot x^\prime A^{-1} x}{(x^\prime x)^2} \leq \frac{(\lambda_1 + \lambda_n)^2}{4 \lambda_1 \lambda_n}. \quad (1.3)$$


\textsuperscript{∗}Corresponding author.
\textit{E-mail addresses:} shuangzhe.liu@canberra.edu.au (S. Liu), luchy@shfc.edu.cn (C. Lu), simo.puntanen@uta.fi (S. Puntanen).

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